Improving Traffic Flows at No Cost

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Abstract

The standard model of traffic flow used in the analysis of urban traffic is the Wardrop equilibrium. The existence of traffic flows that reduce costs for some travelers without increasing the costs for any other travelers when compared to the equilibrium defines a Generalized Braess Paradox. We provide a practical methodology for detecting such flows and report the existence of such a flow in the Sioux Falls study network.

Keywords Multicommodity Traffic, Noncooperative Equilibrium, Nonlinear Programming, Braess Paradox.

1 Introduction

Traffic congestion is becoming a more and more pressing issue for society and a major concern for urban planners. In 1968, Braess [6] identified the possibility that more roads can make traffic worse. In this paper, we take an "inverse" view, that is, that fewer roads, or more-restricted roads, can make traffic better. Specifically, we look for situations in which the total cost of congestion is reduced at negligible additional cost to any traveler. We provide a methodology for identifying such situations and demonstrate that the Sioux Falls study network is an example in which restricting traffic on certain links leads to 33% lower travel times for some travelers while increasing other travelers' times by no more than a quarter of one percent.

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In urban road networks, individual travelers decide on their own travel routes on the basis of factors such as time, cost, convenience. Since they are not acting cooperatively, it is not surprising that these individually chosen routes are not best from society's point of view. In this paper we show how to detect cases where redirecting traffic flows reduces the travel time for some travelers while not increasing travel time for any travelers. Since this redirection can be enforced by restricting access to certain links in the network or by imposing tolls, it is possible to improve society's traffic costs while costs to other individual travelers are reduced or remain the same.

The standard model of traffic flow is that travelers distribute themselves according to Wardrop's user-equilibrium principle (See [12], [16]). This principle states that all used paths between an origin-destination pair will have the same cost, which is no more than the cost on any unused path. Cost is measured as time or some combination of tolls, time, and other factors. Braess [6] used this model to construct a seemingly paradoxical example in which adding a link to a simple network results in a user-equilibrium distribution of flows that is worse for all travelers than the network without the added link. One can also view Braess's network example with the added link as an example in which a nonequilibrium flow (with no flow on the added link) reduces costs for all travelers.

In a previous paper [9], the authors defined a Generalized Braess Paradox to occur whenever there is an alternative distribution of flows which makes some travelers better off and none worse off than in the Wardrop equilibrium distribution. In game-theoretic terms, a Generalized Braess Paradox occurs whenever the user equilibrium is not strongly Pareto optimal. In this paper we show how to detect a Generalized Braess Paradox and report the detection of a Generalized Braess Paradox in the widely known Sioux Falls study network. We thus demonstrate the feasibility of detecting opportunities in which society can improve its total costs without increasing the cost to any individual travelers. The procedure that we develop will also detect occurrences of the "classic" Braess Paradox, in which removing a link results in improved travel cost.

This work is related to, but distinct from, work on finding system-optimal flows in a network. A system-optimal flow in a traffic network minimizes the sum of the costs of all travelers. A system-optimal flow is desirable from society's point of view because it minimizes consumption of resources and production of pollution. The system-optimal flow is usually distinct from the user-equilibrium flow, but typically will require some travelers to incur higher travel costs than in the equilibrium flow. Braess's example is one in which the system-optimal distribution demonstrates the existence of a Generalized Braess Paradox. This, however, is unusual. In more usual cases (See [9]), the system-optimal distribution makes some travelers worse off than in the equilibrium distribution, even when a Generalized Braess Paradox exists. Finding a distribution that demonstrates the existence of a Generalized Braess Paradox is significantly more difficult than finding a system-optimal distribution.

In [9], we showed that a Generalized Braess Paradox can be characterized in terms of a mathematical program. In this paper, we use that characterization to develop a method for detecting the occurrence of a Generalized Braess paradox. We make the mathematical program of [9] tractable by relaxing its constraints to obtain a convex mathematical programming problem that will detect occurrences of the Generalized Braess Paradox. However, due to the particular structure of the relaxed problem, first-order optimality conditions may not hold for the optimal solution, thus rendering inapplicable any algorithm based on standard first-order conditions. Therefore we adopt a special method to solve the problem. The method first uses a sequence of linear programs to identify which nonlinear constraints always hold as equalities, and then whether a Generalized Braess Paradox exists. The number of linear programs is no more than the number of links in the network and usually much less. We apply the method to two small examples and to the well-known Sioux Falls study network with 24 nodes, 76 links, and 528 origin-destination pairs. A Generalized Braess Paradox is found to occur in the Sioux Falls study network. The second of the small examples illustrates that the first-order optimality conditions (Karush-Kuhn-Tucker conditions) cannot be expected to hold for the optimal solution to the relaxed mathematical program, even though it is a convex nonlinear programming problem.

2 Notation and Definition of the Equilibrium Problem

We consider a transportation network with multiple origin-destination (o-d) pairs. Depending on circumstances, demand (usually given as a trip table, specifying for each origin-destination pair the volume per unit time of travelers desiring to move between that origin and destination) may be either elastic or fixed. For the purposes of this paper we assume fixed demand, but note that if demand is elastic, our results hold for the equilibrium demand levels.

We make the following two assumptions. Neither is restrictive in that if either fails to hold, existing methods in the literature [1] can be used to reduce these cases to situations satisfying the assumptions.

1. Travel costs are *additive*, that is, the travel cost of a route is the sum of the traversal costs of the links on the route.

2. The cost of traversing a link is the same for all travelers, and the cost depends on the vector of total link flows, where the *total flow on a single link is the sum of the individual flows on the link between each of the origin-destination pairs.*

An *equilibrium distribution* of flows is a distribution of flows that meets demands and satisfies Wardrop's User-Equilibrium Principle, i.e., every used path between an o-d pair must have the same cost, and all unused paths between the same o-d pair must have cost greater than or equal to that of the used paths. A Wardrop equilibrium corresponds, in a game-theoretic framework, to a (noncooperative) Nash equilibrium. [12]

For a Wardrop equilibrium to be reached, one must assume that travelers have perfect information about travel costs and act to minimize their individual travel costs. Although this may seem to be a strong assumption, most models used in traffic network analysis and planning assume that traffic will be distributed according to a Wardrop equilibrium.

2.1 Notation

As is common in traffic flow theory, our model is built on a network structure with travel costs on each link and known supplies and demands for each node. Our notation accounts for the network structure, properties of links, and properties of the travelers using the network.

Table 1 summarizes the notation we will use. We explain some aspects of the notation.

The elements $a_{i,k}$ of the node-link incidence matrix $\mathbf{A}$ are defined by

$$a_{i,k} = \begin{cases} 1 & \text{if link } k \text{ is directed out of node } i \\ -1 & \text{if link } k \text{ is directed into node } i \\ 0 & \text{otherwise.} \end{cases}$$

Table 1: Notation

k	a link
$t(k)$	the node that link k is directed out of
$h(k)$	the node that link k is directed into
i	a node
d	a destination node
$\mathcal{N}$	the set of nodes in the network
$\mathcal{A}$	the set of links in the network
$\mathbf{A}$	the $\lvert\mathcal{N}\rvert \times \lvert\mathcal{A}\rvert$ node-link incidence matrix of the network
$\mathcal{D}$	the set of destination nodes
b_i^d	for $i \neq d$, the demand for travel from node i to destination d
b_d^d	the negative of the sum of all demands for travel to destination d
$\mathcal{O}^d$	the set of nodes with positive demand for travel to destination d
x_k^d	the amount of flow on link k destined for d
$\mathbf{x}^d$	the vector of link flows destined for d
x_k	the total flow on link k
$\mathbf{x}$	the vector of total link flows
u_i^d	a price (or potential) for node i associated with destination d
$\mathbf{u}^d$	the vector of node prices associated with destination d
z_k^d	a surplus quantity associated with link k and destination d
$\mathbf{z}^d$	the vector of link surpluses associated with destination d
$F_k(\mathbf{x})$	the traversal cost for link k of one unit of flow
$\mathbf{F}(\mathbf{x})$	the vector of link costs
$\bar{x}_k^d$	the equilibrium solution flow on link k destined for d
$\bar{x}_k$	the total flow on link k in the equilibrium solution
$\bar{u}_i^d$	the equilibrium cost of traveling from node i to destination d
$\bar{z}_k^d$	the reduced cost $F_k(\bar{\mathbf{x}}) - \bar{u}_{t(k)}^d + \bar{u}_{h(k)}^d$

In describing the flow on the network, we partition travelers according to their destination. In our previous work, we partitioned travelers according to both their origin and destination. The latter approach is conceptually easier; however, from a computational point of view, the smaller number of classes of travelers is desirable, and does not lose any generality. It is well known (LeBlanc, [11]) that it is not necessary to discriminate between travelers starting from different origins if they are bound for the same destination, or equivalently, that it is not necessary to discriminate between travelers bound for different destinations if they have all started at the same origin.

2.2 The Equilibrium Problem

The Wardrop equilibrium solution can be defined in several equivalent ways (See [12, 14].), e.g., as a variational principle, as the solution of an optimization problem, etc. The particular formulation chosen turns out to be critical in developing a tractable characterization of the Generalized Braess Paradox. For this purpose, we use a Lagrange multiplier definition. The equilibrium problem can be expressed as seeking a solution to

(EQ)

$$\mathbf{F}(\mathbf{x}) - \mathbf{A}^T\mathbf{u}^d - \mathbf{z}^d = \mathbf{0} \quad \forall\, d \in \mathcal{D} \tag{1}$$

$$\mathbf{A}\mathbf{x}^d = \mathbf{b}^d \quad \forall\, d \in \mathcal{D} \tag{2}$$

$$\mathbf{x} = \textstyle\sum_{d\in\mathcal{D}} \mathbf{x}^d \tag{3}$$

$$\textstyle\sum_{d\in\mathcal{D}} \mathbf{z}^d \cdot \mathbf{x}^d = 0 \tag{4}$$

$$\mathbf{x}^d \geq \mathbf{0} \quad \forall\, d \in \mathcal{D} \tag{5}$$

$$\mathbf{z}^d \geq \mathbf{0} \quad \forall\, d \in \mathcal{D} \tag{6}$$

$$u^d_d = 0 \quad \forall\, d \in \mathcal{D} \tag{7}$$

Equation sets (1) and (4) state that on a link with positive flow destined for destination d, the cost of travel on the link k, $F_k(\mathbf{x})$, is equal to the price difference, u^d_i-u^d_j, corresponding to destination d, between the two end nodes, i and j, of the link. If there is no flow directed towards d on link k, then (4) allows z^d_k to be positive and the cost of travel on link k may be greater than or equal to the difference in prices. Equation set (2) requires that flows directed toward d satisfy demand at origin and destination nodes and conserve flow at other nodes. Equation (3) defines the total flow on a link to be the sum of the flows on that link headed to the different destinations. There is always one node price u_i^d for each d that is arbitrary. Equation set (7) removes this ambiguity by defining the price at the destination nodes to be zero. An equilibrium solution, denoted $\{(\bar{\mathbf{x}}^d, \bar{\mathbf{u}}^d, \bar{\mathbf{z}}^d)\}_{d \in \mathcal{D}}$, is a solution to equations (1)-(7). The node price $\bar{u}_i^d$ of an equilibrium solution becomes the cost of traveling from node i to destination d along links with $\bar{x}_k^d > 0$.

3 The Existence of Improved Flows

Given a Wardrop equilibrium set of flows, we wish to determine whether there is another distribution of flows that makes some travelers better off and no travelers worse off than in this equilibrium. To that end, we define a nonlinear program which minimizes system cost subject to the constraint that no traveler has cost greater than in the given equilibrium. The constraints are similar to those of the equilibrium problem, (EQ), except that instead of requiring that the traversal cost on a used link equal the price difference of its nodes, we allow the traversal cost of the link to be less than or equal to the price difference of its nodes. In this way, the formulation allows nonequilibrium flows.

For given $\mathbf{A}$, $\mathbf{F}$, demand vectors $\mathbf{b}^d$, and equilibrium travel costs $\bar{u}_s^d$, define the following optimization problem originally introduced in [9], which we henceforth call the ***Equilibrium Improvement Problem***, (EIP). (In [9], we referred to this as the Braess Optimization Problem.)

(EIP)

$$
\begin{array}{rcll}
\min & \mathbf{x} \cdot \mathbf{F}(\mathbf{x}) & & \\
\text{subject to} & \mathbf{F}(\mathbf{x}) - \mathbf{A}^T \mathbf{u}^d - \mathbf{z}^d \leq \mathbf{0} & \forall\, d \in \mathcal{D} & (8) \\
& \mathbf{A}\mathbf{x}^d = \mathbf{b}^d & \forall\, d \in \mathcal{D} & (9) \\
& \mathbf{x} = \sum_{d \in \mathcal{D}} \mathbf{x}^d & & (10) \\
& \sum_{d \in \mathcal{D}} \mathbf{z}^d \cdot \mathbf{x}^d = 0 & & (11) \\
& \mathbf{x}^d \geq \mathbf{0} & \forall\, d \in \mathcal{D} & (12) \\
& \mathbf{z}^d \geq \mathbf{0} & \forall\, d \in \mathcal{D} & (13) \\
& u_d^d = 0 & \forall\, d \in \mathcal{D} & (14) \\
& u_s^d \leq \bar{u}_s^d & \forall\, s \in \mathcal{O}^d & (15)
\end{array}
$$

The constraints of (EIP) are very similar to the equilibrium problem (EQ). The differences are:

1. As noted above, Constraint Set (8) is a set of inequalities instead of equations. The inequalities allow nonequilibrium flows.

2. There is an additional constraint set, (15), which forces the travel cost from any origin to destination to be less than or equal to that of the equilibrium flow.

Under the reasonable assumptions that $\mathbf{x}$ is nonnegative and that $\mathbf{F}(\mathbf{x})$ is convex and monotone $((\mathbf{y}-\mathbf{x})\cdot(\mathbf{F}(\mathbf{y})-\mathbf{F}(\mathbf{x}))\geq 0$ for all feasible $\mathbf{x}$, $\mathbf{y}$ [10]), the objective function of (EIP) is easily shown to be convex. Thus without constraint (11), (EIP) would be a convex optimization problem.

Any feasible solution to (EIP) with objective function value less than that of the equilibrium flow reduces the travel cost for some travelers, and due to the last set of constraints, does not increase the travel cost for any travelers. Thus if an equilibrium solution is not optimal for (EIP), a Generalized Braess Paradox exists. That the converse also holds when the cost functions F_k are nonnegative and nondecreasing in each of their arguments was shown in [9]. It follows that under these mild conditions on F_k, determining the existence of flows that improve on a Wardrop Equilibrium is equivalent to testing (EIP) to see if a Wardrop equilibrium is optimal. In the following sections we will develop methods to test optimality of the Wardrop equilibrium. We first establish that if there is a feasible solution to (EIP) for which some constraint corresponding to a used link in set (8) holds strictly, then the equilibrium solution is not optimal for (EIP) and a Generalized Braess Paradox exists.

Proposition 1 *If there exists a feasible solution to* (EIP) *with the property that for some link k and destination d,*

$$x_k^d > 0 \textit{ and } F_k(\mathbf{x}) - u_{t(k)}^d + u_{h(k)}^d < 0,$$

then a Generalized Braess Paradox exists.

Proof: Suppose that the triples $(\mathbf{x}^d, \mathbf{u}^d, \mathbf{z}^d)$ define a feasible solution to (EIP) and there exists a link k^* and destination d^* such that

$$x_{k^*}^{d^*} > 0 \text{ and } F_{k^*}(\mathbf{x}) - u_{t(k^*)}^{d^*} + u_{h(k^*)}^{d^*} < 0.$$

From constraint set (15), we know that no traveler is worse off than in equilibrium. Since $x_{k^*}^{d^*} > 0$, there exists an origin s^* which contributes flow to link k^* that is destined for d^*; more specifically, there is a path P of links k joining s^* to d^* such that $k^* \in P$ and $x_k^{d^*} > 0$ for all links $k \in P$. Since $x_k^{d^*} > 0$ for these links, $z_k^{d^*} = 0$ on these links. Then for each of these links, Constraint Set (8) gives

$$F_k(\mathbf{x}) \leq u_{t(k)}^{d^*} - u_{h(k)}^{d^*}.$$

Our assumption of strict inequality gives

$$F_{k^*}(\mathbf{x}) < u_{t(k^*)}^{d^*} - u_{h(k^*)}^{d^*}.$$

Summing over $k \in P$, and using Constraint Sets (14) and (15) we have

$$\sum_{k \in P} F_k(\mathbf{x}) < u_{s^*}^{d^*} \leq \bar{u}_{s^*}^{d^*},$$

Thus we have travelers using path P to go from s^* to t^* with a lower cost than in equilibrium. ∎

4 A Computational Approach for Local Improvements

(EIP) provides a direct method of checking for the existence of a Generalized Braess Paradox by solving an optimization problem. However, for even moderately large networks (EIP) is difficult to solve because the complementarity constraint (11), which essentially defines for each destination d the subnetwork of arcs that may be used by flows destined for d, is not convex. Solving (EIP) implies the need to (implicitly or explicitly) enumerate all feasible subnetworks of the network. Since for each destination, flows may use a different subnetwork, solving (EIP) may require an extremely large enumeration. We therefore treat a more tractable version of the problem for which we can detect many instances of the Generalized Braess Paradox using a finite sequence of linear programs.

In order to develop the more tractable test, we replace the troublesome Constraint (11) with a more restrictive, but more tractable, condition. This new problem will identify a local Generalized Braess Paradox, in the sense that our search for an improved flow is restricted to using essentially the same set of links used in the Wardrop equilibrium solution. The existence of a solution of the more restrictive problem that has a lower objective function value than the equilibrium solution will guarantee the existence of a Generalized Braess Paradox. However, because the problem is more restrictive, an improved solution using a different subnetwork may remain undetected. Therefore even when the equilibrium solution is optimal for the modified problem, a Generalized Braess Paradox may exist as shown by Example 1 of [9]. This limitation is shared by all tests for the Braess Paradox of which we are aware ([7], [15]), in that none will detect a Braess Paradox that uses flows on a subnetwork distinct from that of the equilibrium solution.

The ***Restricted Equilibrium Improvement Problem***, (R-EIP), is

(R-EIP)

$$
\begin{array}{llll}
\min & \mathbf{x} \cdot \mathbf{F}(\mathbf{x}) & & \\
\text{subject to} & \mathbf{F}(\mathbf{x}) - \mathbf{A}^T \mathbf{u}^d - y\bar{\mathbf{z}}^d \leq \mathbf{0} & \forall\ d \in \mathcal{D} & (16) \\
& \mathbf{A}\mathbf{x}^d = \mathbf{b}^d & \forall\ d \in \mathcal{D} & (17) \\
& \mathbf{x} = \sum_{d \in \mathcal{D}} \mathbf{x}^d & & (18) \\
& \sum_{d \in \mathcal{D}} \bar{\mathbf{z}}^d \cdot \mathbf{x}^d = 0 & & (19) \\
& \mathbf{x}^d \geq \mathbf{0} & \forall\ d \in \mathcal{D} & (20) \\
& u_d^d = 0 & \forall\ d \in \mathcal{D} & (21) \\
& u_s^d \leq \bar{u}_s^d & \forall\ s \in \mathcal{O}^d & (22)
\end{array}
$$

This formulation entails two changes from (EIP). Constraint Set (8) has been replaced with Constraint Set (16). Since the variable y can be set to a very large number, when the equilibrium value $\bar{z}_k^d > 0$, the corresponding constraint is vacuous just as is the case of Constraint Set (8) when $z_k^d > 0$. Constraint (11) has been replaced with Constraint (19), which requires that $x_k^d = 0$ whenever $\bar{z}_k^d > 0$. Thus any feasible solution to (R-EIP) has flow going to destination d only on links k with $\bar{z}_k^d = 0$.

If $\mathbf{F}$ is convex and monotone, the objective function of (R-EIP) is convex and (R-EIP) is a convex optimization problem. All constraints except those involving $\mathbf{F}$ are linear. Our aim is to determine if the Wardrop equilibrium solution $\{(\bar{\mathbf{x}}^d, \bar{\mathbf{u}}^d, \bar{\mathbf{z}}^d)\}_{d \in \mathcal{D}}$ is optimal for (R-EIP). If a constraint qualification held for the problem, one might use the first-order necessary (KKT) conditions. However, as shown by Example 2 in Section 5, the KKT conditions for (R-EIP) do not necessarily hold for the equilibrium solution, even when it is optimal. Therefore no constraint qualification can be assumed to hold for the problem, and methods other than those based on the standard first-order conditions must be used.

Convex programming problems for which no constraint qualification holds have been studied extensively by Ben-Israel, Ben-Tal and Zlobec [5]. Using their approach and an algorithm proposed by Kerzner [2], we first determine if one or more nonlinear constraints hold as strict inequalities for some feasible solution. If even one such (nonvacuous) constraint exists, Proposition 1 states that there is a Generalized Braess Paradox. If it is determined that no such constraint exists, we formulate a single linear program that searches for a feasible direction of improvement. The existence of such a direction will establish the existence of a Generalized Braess Paradox. If no such direction exists, then the Wardrop equilibrium solution is optimal for (R-EIP).

Kerzner's algorithm for finding the constraints that can be satisfied strictly solves a sequence of linear programs. The number of linear programs that must be solved is no more than the number of constraints and usually far less. For example, model (R-EIP) for the Sioux Falls study network has 1655 nonlinear constraints, but requires the solution of only three linear programs to determine which constraints can be satisfied strictly. The algorithm as adapted for (R-EIP) is described in the appendix.

5 Computational Results

We consider three examples in detail. The first two use the five-link bridge network studied by Braess [6] to illustrate his paradox, and the third is the well-known Sioux Falls study network [3]. The first of the small examples is a straightforward application of the method as described in the appendix. The second example on the same network is a case in which there is no Generalized Braess Paradox, that is, the equilibrium solution is optimal for (EIP), but the Karush-Kuhn-Tucker conditions do not hold. The data, models and numerical results for all three examples are given in [8].

Example 1

The network for Example 1 is shown in Figure 1. The demand for travel between the origin s and the destination t is 6 units of flow. The cost functions for the five links, the equilibrium solution, the system optimal solution, and an improved solution illustrating a Generalized Braess Paradox are shown in Table 2. There is no classic Braess Paradox for this problem because, as is

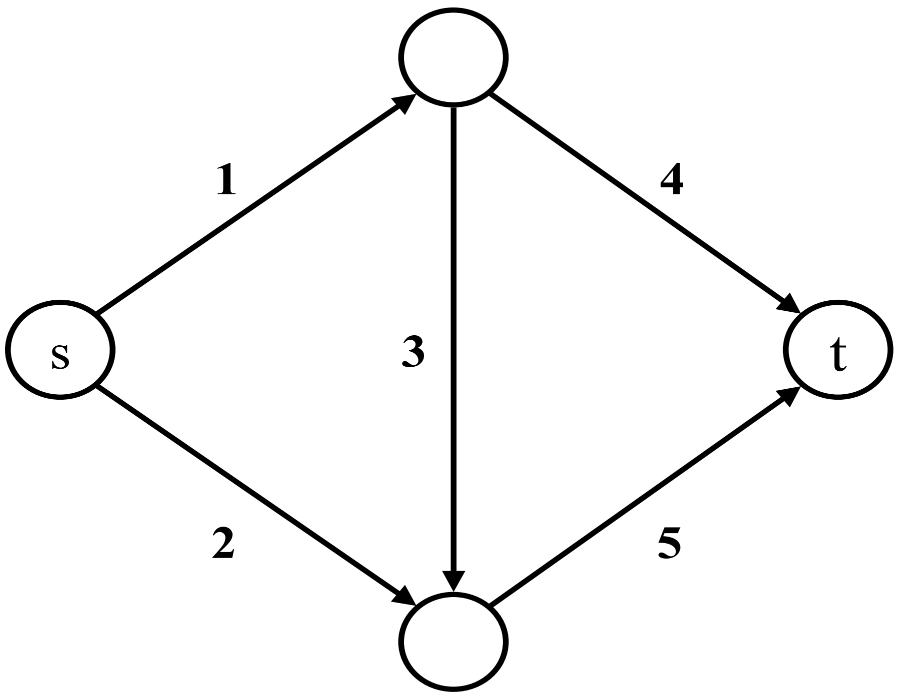

Figure 1: Bridge Network

Table 2: Data for Example 1, with a Generalized Braess Paradox but no Classic Braess Paradox

Link	Cost Function	Equilibrium	System Optimal	(R-EIP) Optimal	Equilibrium without Link 3
		Link Flows			
1	$1.4x_1$	4.000	3.630	3.464	2.914
2	$5.4 + \sqrt{4x_2^2 + 9}$	2.000	2.370	2.536	3.086
3	$2.4x_3$	2.000	1.010	1.011	0.000
4	$7.8 + \sqrt{4x_4^2 + 9}$	2.000	2.620	2.453	2.914
5	$2x_5$	4.000	3.380	3.547	3.086
Route		Route Costs			
1, 4		18.400	18.920	18.400	18.434
1, 3, 5		18.400	14.266	14.372	
2, 5		18.400	17.770	18.387	18.434
Most Costly Used Route		18.400	18.920	18.400	18.434
System Cost		110.400	106.093	106.293	110.607

Table 3: Data for Example 2, with No Generalized Braess Paradox

Link	Cost Function	Equilibrium	System Optimal
1	$1.6x_1$	4.00	3.52
2	$5.4 + \sqrt{4x_2^2 + 9}$	2.00	2.48
3	$2x_3$	2.00	1.04
4	$5.4 + \sqrt{4x_4^2 + 9}$	2.00	2.48
5	$1.6x_5$	4.00	3.52
Highest Used Route Cost		16.80	17.99
System Cost		100.8	97.3

also shown in Table 2, eliminating link three does not result in an improved equilibrium travel cost from node 1 to node 4. Because the problem is small, it is a simple matter to solve (EIP) or (R-EIP) directly to find a Generalized Braess Paradox if one exists. The (R-EIP) optimal solution shown in Table 2 reduces the travel cost for travelers using the route consisting of links 1, 3, and 5 by 22 percent, and does not increase the cost for any other travelers, thus establishing the existence of a Generalized Braess Paradox. In this particular example, any convex optimizer can be counted on to give a correct solution to (R-EIP) because the nonlinear constraints can be satisfied strictly for some feasible solution. The details of our general approach as applied to this example are given in the appendix.

Example 2

The network for Example 2 is the same simple network as for Example 1, with the same demand for travel. The cost structure has been changed to eliminate the occurrence of a Generalized Braess Paradox. The equilibrium solution is optimal for (R-EIP). However, the first order optimality conditions do not hold at the equilibrium solution. The link costs, the equilibrium solution and the system-optimal solution are shown in Table 3.

Applying the algorithm described in the appendix to (R-EIP), we find at the first iteration that both of the nonlinear constraints (those in (16) corresponding to links 2 and 4) must hold with equality for all feasible solutions of (R-EIP). We then conclude (see the appendix) that the flows on links 2 and 4 are constant for all feasible solutions of (R-EIP). Due to the simple structure of this example, it is immediately apparent that if there are no feasible changes to the flows on links 2 and 4, there are no feasible changes to the flows on the other three links. Thus we know that the equilibrium solution is optimal for (R-EIP) and there is no local Generalized Braess Paradox.

If the example were not so simple, we would proceed by replacing the nonlinear constraints in (R-EIP) by linear constraints defined by replacing the arguments of the strictly convex cost functions by their constant values. The result is a nonlinear program with convex objective function and linear constraints. For this program, we can determine optimality of the equilibrium solution simply by checking for the existence of a feasible direction of decrease of the objective function. This requires only the solution of a single linear program. We would find that there is no such feasible direction of improvement, and we would conclude that the equilibrium solution is optimal for (R-EIP) and there is no local Generalized Braess Paradox. Even though the equilibrium solution is optimal for (R-EIP), the Karush-Kuhn-Tucker conditions do not hold at the equilibrium solution.

Example 3

The Sioux Falls study network is often used as a test network for transportation models. It consists of 24 nodes, 76 one-way links and 528 origin-destination pairs. Thus, most of the nodes are both origins and destinations. The network is shown in Figure 2. The network structure, the trip table specifying required flows, and an accurate equilibrium solution can be found at [3]. The linear programs described in the appendix were formulated using the LINGO modeling language (Details are available at [8]). For Bar-Gera's equilibrium solution [3], the linear programs have approximately 2500 constraints and 4000 variables. The cost function used is the standard fourth-power polynomial used in traffic analysis and is also available at [3].

The standard cost function is meant to be essentially constant in the low-volume free-flow range and to then increase rapidly for flows exceeding a given capacity level. For equilibrium flows that are under half of the given capacity level, we replaced the fourth degree polynomial by a constant function. For travelers on these links, a small change in flows on these links will have a negligible effect on cost. The fact that several of the equilibrium link flows were in this free-flow range turned out to be critical to the detection of the Generalized Braess Paradox. After finding the optimal changes for (R-EIP), we recalculated link costs using the original quartic functions. We found that while reducing some travelers costs by 33%, no link costs increased by more than 0.25%. These increases occur on the arcs that were set to a constant

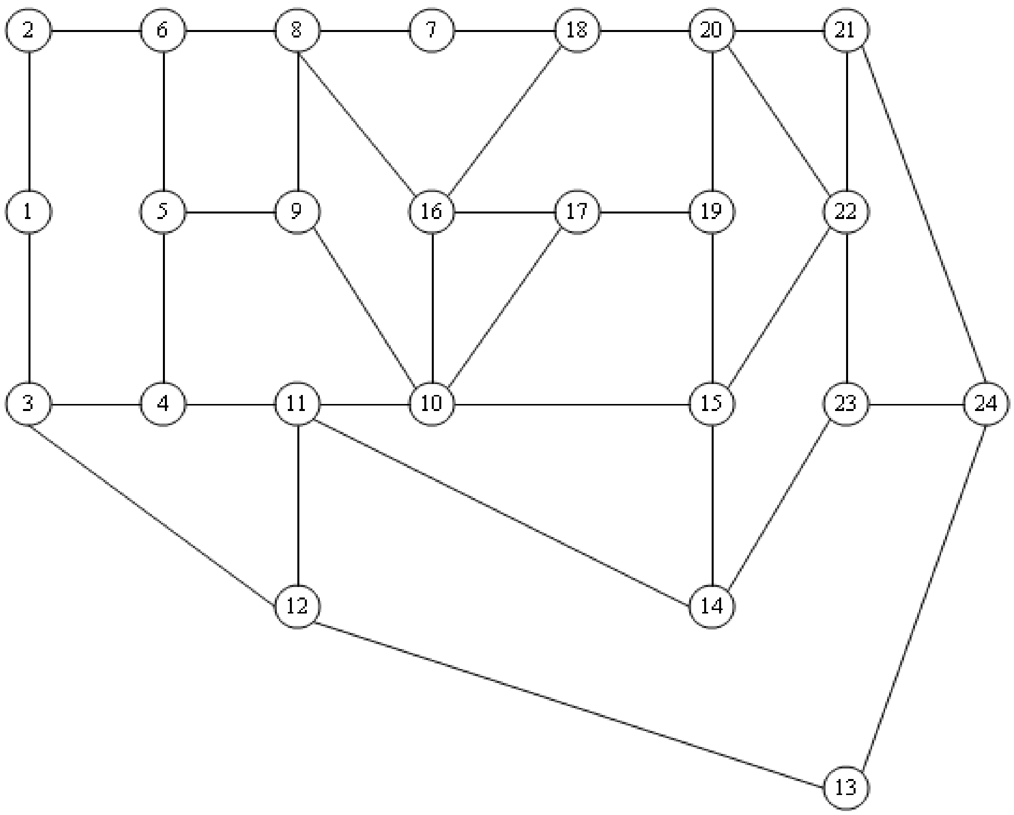

Figure 2: Sioux Falls Network

cost, and the flows on these links were still under 50% of capacity.

After solving the first LP as described in the appendix (see also [8]), we found that 72 of the 76 (one-way) links must have constant cost for all feasible flows. Solving a second LP added two links to the set with constant cost. After solving a third LP, we found that nonlinear constraints corresponding to the last two links [5-6] and [6-5] can be satisfied strictly. All three of the LP's were solved in seconds on a desktop Windows machine using LINGO. The dual prices from the third LP give the changes in destination-based flows that will make links [5-6] and [6-5] have costs that are lower than the price differentials. The links that have changes in flow, and the directions of these changes, are shown in Figure 3. By Proposition 1, these changes demonstrate the occurrence of a Generalized Braess Paradox for the Sioux Falls study network. After determining that a Generalized Braess Paradox exists and that all costs were constant except those corresponding to links [5-6] and [6-5], we found the optimal solution to (R-EIP). The optimal solution involved flow changes to three additional arcs and several additional destinations.

6 Conclusions

The method presented in this paper identifies situations in which, when compared with the Wardrop equilibrium, alternate flows exist that reduce cost for some travelers without increasing cost for any other travelers. The network presented by Braess [6] is an example of such a situation. That the phenomenon can occur in much more complex (nonlinear cost structure, multiple origins and destinations, etc.) situations is shown by the small examples in [9] and, in this paper, by the Sioux Falls study network. Models of urban areas can easily involve thousands of links and origin-destination pairs. Because the method presented involves only the solution of linear programs, we expect that it can be directly applied to large urban networks. When improved flows are found, congestion and societal costs can be reduced, but individual travelers face negligible increases in costs. This is in contrast to system-optimal flows where typically some travelers face significant increases in cost.

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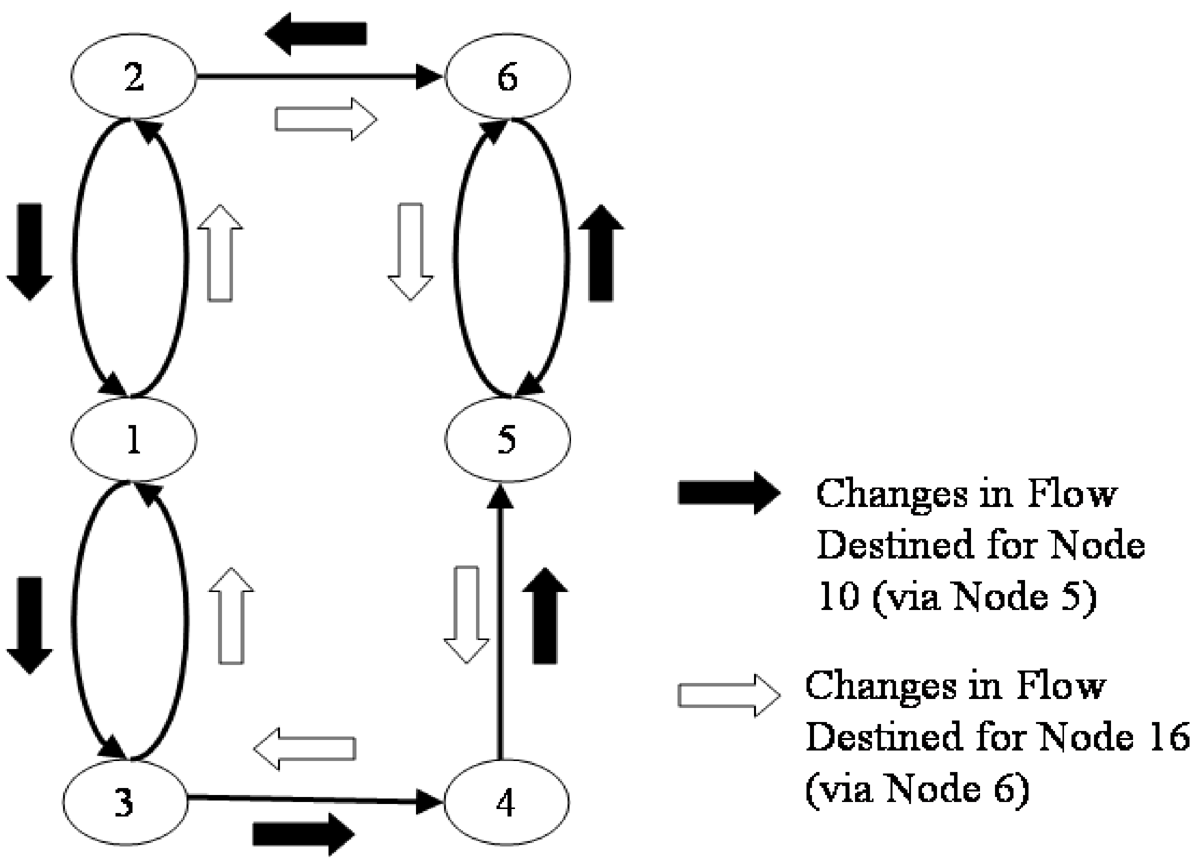

Figure 3: Improving Changes in Flow for Sioux Falls Network

version of LINGO used for the Sioux Falls example and for helping with the implementation.

7 Appendix: Finding the Set of Always Binding Constraints

In this appendix we apply Kerzner's algorithm [2] to (R-EIP). Because (R-EIP) has many sets of constraints and variables, the details become complicated. Therefore we first describe a version of the algorithm for a more general convex program.

We wish to determine whether a vector $\mathbf{x}^*$ is optimal for the following convex programming problem.

(CP)

$$\begin{array}{rll} \min & f(\mathbf{x}) & \\ \text{subject to} & g_i(\mathbf{x}) \leq 0 & i = 1, 2, ...m \\ & \mathbf{Ax} = \mathbf{b} & \end{array}$$

where f and the g_i are differentiable and faithfully convex [13], and $\mathbf{A}$ is a matrix and $\mathbf{b}$ a vector of appropriate dimensions. We also assume, without loss of generality, that at $\mathbf{x}^*$ all of the nonlinear constraints, g_i, are binding. Any constraints that are not binding at $\mathbf{x}^*$ may be ignored for our current purposes.

Let $P^=$ be the set of always binding nonlinear constraints, that is, $P^=$ is the set of nonlinear constraints which are binding for all feasible solutions of (CP). By definition all constraints not in the set $P^=$ can be satisfied strictly for some feasible solution, and by taking a convex combination of such feasible solutions, we may obtain a single feasible solution for which all constraints not in $P^=$ hold strictly.

As will be shown below, if the set $P^=$ is known, the nonlinear constraints in $P^=$ may be replaced by linear constraints, and the remaining nonlinear constraints (those not in $P^=$) will hold strictly for some feasible solution. Thus the Slater constraint qualification [4] will hold, and it will be a simple matter to check the optimality of $\mathbf{x}^*$ by solving a single linear program.

Kerzner's algorithm for finding the set $P^=$ of (R-EIP) incrementally builds up the set of constraints known to be in $P^=$. It starts by assuming $P^=$ is empty, and at each iteration finds at least one more constraint (often many more) that is a member of $P^=$, or determines that $P^=$ is already completely specified, in which case the algorithm terminates.

To start the algorithm, we check for the existence of a feasible direction $\mathbf{d}$ so that all of the nonlinear constraints evaluated at $\mathbf{x}^* + t\mathbf{d}$ for small $t > 0$ will hold strictly, that is, we look for a $\mathbf{d}$ that satisfies

$$\begin{aligned} \nabla g_i(\mathbf{x}^*)^T \mathbf{d} \leq -1 \quad i = 1, 2, \ldots m \\ \mathbf{A}\mathbf{d} = \mathbf{0} \end{aligned}$$

If such a $\mathbf{d}$ exists then $P^=$ is empty and all of the nonlinear constraints can be satisfied strictly and the algorithm terminates. If no such $\mathbf{d}$ exists, then the linear program
(FD-P)

$$\begin{array}{rc} \max & \mathbf{0} \cdot \mathbf{d} \\ \text{subject to} & \nabla g_i(\mathbf{x}^*)^T \mathbf{d} \leq -1 \quad i = 1, 2, \ldots m \\ & \mathbf{A}\mathbf{d} = \mathbf{0} \end{array}$$

has no feasible solution and by the duality theorem of linear programming its dual
(FD-D)

$$\begin{array}{rc} \min & -\sum \alpha_i \\ \text{subject to} & \sum \nabla g_i(\mathbf{x}^*)\alpha_i + \mathbf{A}^T\beta = \mathbf{0} \\ & \alpha \geq \mathbf{0}, \end{array} \qquad (23)$$

must be either infeasible or unbounded. However, $\mathbf{0}$ is a feasible solution of (FD-D), and thus if (FD-P) has no feasible solution, (FD-D) must be unbounded. In that case some feasible solution to (FD-D) has at least one positive component in α.

Let α be a feasible solution of (FD-D) with at least one component, for example the first component, α_1, positive. Suppose that for some feasible solution of (CP), the first nonlinear constraint holds strictly. Then there must exist a feasible direction $\mathbf{d}$ of (CP) satisfying

$$\mathbf{d}^T \nabla g_1 < 0$$
$$\mathbf{d}^T \nabla g_i \leq 0 \quad i \neq 1$$
$$\mathbf{d}^T \mathbf{A}^T = 0$$

Now "left" multiply the constraint (23) of (FD-D) by this feasible direction vector $\mathbf{d}$. The first summand of the left hand side consists of the positive α_1 times the (negative) inner product $\mathbf{d}^T \nabla g_1$. Thus the first summand is negative. Similarly all others are nonpositive. Noting that $\mathbf{d}^T \mathbf{A}^T \beta = 0$, we see that the left hand side is negative and the right hand side is zero - a contradiction. Therefore we conclude that whenever an α_i is positive the corresponding $\mathbf{d}^T \nabla g_i$ cannot be negative. Hence the corresponding constraints of (CP) can never hold strictly and are members of $P^=$. Thus if we solve (FD-D), and find one or more positive α_i, we know that the corresponding constraints are members of $P^=$.

The next step is to replace nonlinear constraints known to be members of $P^=$ with linear constraints. The assumption of faithful convexity means that $g_i(\mathbf{x})$ in $P^=$ can be broken into linear and strictly convex parts. Since by definition a constraint belonging in $P^=$ is constant on the feasible set, the strictly convex part of the constraint and the linear part must each be constant on the feasible set. It then follows that the argument of the strictly convex part of the function must be constant on the feasible set. Therefore we can replace the nonlinear constraint by linear constraints which require i) that the argument of the strictly convex part of the constraint equal its unique value on the feasible set, and ii) that the linear part of the constraint equal its unique value on the feasible set. The result is that we can write (CP) as a convex program with at least one fewer nonlinear constraint.

We repeatedly apply the above method until all constraints are determined to be in $P^=$ and have been replaced by linear constraints, or the only solutions of (FD-D) have $\alpha = 0$. When the only solutions of (FD-D) have $\alpha = 0$, all of the remaining nonlinear constraints hold strictly for some feasible solution of (CP), which by Proposition 1 means that a Generalized Braess Paradox exists. If all nonlinear constraints are in $P^=$, we will have reduced the (CP) to a problem with a convex objective function and linear constraints. The optimality of $\mathbf{x}^*$ may then be determined by solving a single linear program, e.g., by seeking a feasible direction of descent of the objective function.

As an illustration we apply the above method to Example 1 of Section 5. The network is shown in Figure 1, and the link cost functions are given in Table 2. The equilibrium solution for a total flow of 6 units is shown in Table 2, as are the system optimal flows and improved flows to be determined by the above method.

First formulate the convex program (R-EIP) using the data from Figure 2. Because the equilibrium solution has flow on all links, **z** is equal to zero and may be omitted from the formulation. As there is only one destination, we omit the superscripts from the formulation.

(R-EIP)

$$\min 1.4x_1^2 + 5.4x_2 + x_2\sqrt{4x_2^2+9} + 2.4x_3^2 + 7.8x_4 + x_4\sqrt{4x_4^2+9} + 2x_5^2$$

$$\begin{aligned} st \qquad 1.4x_1 + u_2 - u_s &\leq 0 \\ 5.4 + \sqrt{4x_2^2+9} + u_3 - u_s &\leq 0 \\ 2.4x_3 + u_3 - u_2 &\leq 0 \\ 7.8 + \sqrt{4x_4^2+9} + u_t - u_2 &\leq 0 \\ 2x_5 + u_t - u_3 &\leq 0 \\ x_1 + x_2 &= 6 \\ -x_1 + x_3 + x_4 &= 0 \\ -x_2 - x_3 + x_5 &= 0 \\ -x_4 - x_5 &= -6 \\ x_j &\geq 0 \quad j = 1, 2, ..5 \\ u_t &= 0 \\ u_s &\leq 18.4 \end{aligned}$$

Next we formulate the primal feasible direction problem (FD-P) which searches for a direction, **d**, that will make the nonlinear constraints hold strictly.

(FD-P)

$$\begin{aligned} 1.4d_{x_1} + d_{u_2} - d_{u_s} &\leq 0 \\ 1.6d_{x_2} + d_{u_3} - d_{u_s} &\leq -1 \\ 2.4d_{x_3} + d_{u_3} - d_{u_2} &\leq 0 \\ 1.6d_{x_4} + d_{u_t} - d_{u_s} &\leq -1 \end{aligned}$$

$$\begin{aligned} 2d_{x_5} + d_{u_t} - d_{u_3} &\le 0 \\ d_{x_1} + d_{x_2} &= 0 \\ -d_{x_1} + d_{x_3} + d_{x_4} &= 0 \\ -d_{x_2} - d_{x_3} + d_{x_5} &= 0 \\ -d_{x_4} - d_{x_5} &= 0 \\ d_{u_t} &= 0 \\ d_{u_s} &\le 0 \end{aligned}$$

The existence of a solution to this set of inequalities is equivalent to the Slater Condition, which requires that the nonlinear constraints be satisfied strictly at some feasible point. Although the Slater Condition will turn out to hold for this problem, in general it cannot be expected to hold, and, in fact, it does not hold in Examples 2 and 3. We first convert the above set of inequalities into a linear program by forming an objective function with cost coefficients all equal to zero. Then form the dual of the linear program which we denote (FD-D). We denote the dual variables corresponding to the nonlinear constraints of (R-EIP) (second and fourth constraints) by α_j for $j = 2, 4$ and those corresponding to the linear constraints by β_j for all other values of j.

(FD-D)

$$\begin{array}{rc} \max & \alpha_2 + \alpha_4 \\ st & 1.4\beta_1 + \beta_6 - \beta_7 = 0 \\ & 1.6\alpha_2 + \beta_6 - \beta_8 = 0 \\ & 2.4\beta_3 + \beta_7 - \beta_8 = 0 \\ & 1.6\alpha_4 + \beta_7 - \beta_9 = 0 \\ & 2\beta_5 + \beta_8 - \beta_9 = 0 \\ & \beta_1 + \alpha_2 + \beta_{11} = 0 \\ & -\beta_1 + \beta_3 + \alpha_4 = 0 \\ & -\alpha_2 - \beta_3 + \beta_5 = 0 \\ & -\alpha_4 - \beta_5 + \beta_{10} = 0 \\ & \alpha_2, \alpha_4, \beta_1, \beta_3, \beta_5, \beta_{11} \ge 0 \end{array}$$

As pointed out above, (FD-D) is either unbounded or it has an optimal solution with optimal objective function value equal to zero. We find that the optimal solution of (FD-D) has $\alpha_2 = 0$ and $\alpha_4 = 0$. Therefore, we know that (FD-P) has an optimal solution which is a feasible direction leading to a point that satisfies both nonlinear constraints of (R-EIP) strictly. From Proposition 1 it follows that a Generalized Braess Paradox exists.

The optimal solution to (FD-P), that is, the direction vector leading to an interior point, may be obtained from the dual prices of (FD-D). An improved flow for the original network, demonstrating the Generalized Braess Paradox, may then be obtained by moving in this direction. The optimal solution to (R-EIP) is shown in Table 2. It is very close to the solution obtained by moving in the direction given by the solution to (FD-P). This solution reduces travel cost by 22% for travelers moving along the path defined by links 1, 3, and 5, and does not increase travel cost for any travelers when compared with the equilibrium solution.

We end this appendix with general formulations of the (R-EIP), (FD-P) and (FD-D) for Kerzner's algorithm. We make one change of notation for the general formulation. In this appendix we have used $\mathbf{d}$ to indicate a direction vector in the feasible set as in common in optimization theory. However, in our general formulation of the equilibrium problem (EQ) and equilibrium improvement problem (EIP), d is used to index the destination nodes of the network. We return to this use of d, and now represent a component of a direction vector by putting an arrow over the variable corresponding to that component. Thus the component of a feasible direction corresponding to x_1 is denoted $\vec{x}_1$.

Under the assumption that $F_k(\mathbf{x}) = f_k(x_k)$ the individual cost constraints of (R-EIP) have the form

$$f_k(x_k) - u^d_{t(k)} + u^d_{h(k)} - \bar{z}^d_k y \leq 0.$$

for each link k and destination d. Note that for each link, that is for fixed k, the constraints corresponding to the various destinations d all have the same cost function f_k. For the constraints with strictly convex cost functions, if it is known that for some destination d, the constraint indexed by (k, d) belongs to $P^=$, the unique feasible value of f_k will be known, and the nonlinear function can be replaced by a constant in the cost constraints of all destinations involving that link. Thus due to the special structure of (EIP) and (R-EIP), at each iteration of Kerzner's algorithm we are able to remove many more nonlinear constraints than in the general case.

To make the preceding precise, first define $\mathcal{L}$ to be the set of arcs k for which f_k is affine. Then define, $\mathcal{P} \subseteq \mathcal{A} - \mathcal{L}$ to be the set of links with nonlinear costs such that it is known that for some destination, the corresponding cost constraint belongs to $P^{=}$. At each iteration of the algorithm more links will be included in $\mathcal{P}$ until it is as large as possible.

Because the constraints of (R-EIP) do not allow flow on links with $\bar{z}_k^d > 0$ for all $d \in \mathcal{D}$, assume that we have deleted all such links from the network. To apply the algorithm in the general case, we first rewrite (R-EIP) to explicitly separate the cost constraints into three groups, namely those with affine cost functions, that is, $k \in \mathcal{L}$, those with nonlinear cost functions that are known to have a single value for all feasible solutions, that is, $k \in \mathcal{P}$, and the remaining constraints. Then (R-EIP) can be reformulated as

$$\begin{array}{lll}
\min & \mathbf{x} \cdot \mathbf{F}(\mathbf{x}) & \\
\text{subject to} & f_k(\bar{x}_k) - u^d_{t(k)} + u^d_{h(k)} - \bar{z}^d_k y \leq 0 & \text{if } k \in \mathcal{P}, \forall\ d \in \mathcal{D} \\
& f_k(x_k) - u^d_{t(k)} + u^d_{h(k)} - \bar{z}^d_k y \leq 0 & \text{if } k \in \mathcal{L}, \forall\ d \in \mathcal{D} \\
& f_k(x_k) - u^d_{t(k)} + u^d_{h(k)} - \bar{z}^d_k y \leq 0 & \text{if } k \notin \mathcal{P} \cup \mathcal{L}, \forall\ d \in \mathcal{D} \\
& \mathbf{A}\mathbf{x}^d = \mathbf{b}^d & \forall\ d \in \mathcal{D} \\
& -\sum_{d \in \mathcal{D}} x^d_k + \bar{x}_k = 0 & \text{if } k \in \mathcal{P} \\
& -\sum_{d \in \mathcal{D}} x^d_k + x_k = 0 & \text{if } k \notin \mathcal{P} \\
& \sum_{d \in \mathcal{D}} \bar{\mathbf{z}}^d \cdot \mathbf{x}^d = 0 & \\
& \mathbf{x}^d \geq \mathbf{0} & \forall\ d \in \mathcal{D} \\
& u^d_d = 0 & \forall\ d \in \mathcal{D} \\
& u^d_s \leq \bar{u}^d_s & \forall\ s \in \mathcal{O}^d
\end{array}$$

The linear program that searches for an interior direction is

(FD-P)

$$\begin{array}{llll}
\min & 0 & & \\
\text{subject to} & \vec{u}^d_{t(k)} - \vec{u}^d_{h(k)} + \bar{z}^d_k \vec{y} \geq 0 & \text{if } k \in \mathcal{P} & \\
& -\frac{\partial f_k(\bar{x}_k)}{\partial x_k} \vec{x}_k + \vec{u}^d_{t(k)} - \vec{u}^d_{h(k)} + \bar{z}^d_k \vec{y} \geq 0 & \text{if } k \in \mathcal{L} & (24) \\
& -\frac{\partial f_k(\bar{x}_k)}{\partial x_k} \vec{x}_k + \vec{u}^d_{t(k)} - \vec{u}^d_{h(k)} + \bar{z}^d_k \vec{y} \geq 1 & \text{if } k \notin \mathcal{P} \cup \mathcal{L} & \\
& \mathbf{A}\vec{\mathbf{x}}^d = \mathbf{0} & \forall\ d \in \mathcal{D} & (25) \\
& -\sum_{d \in \mathcal{D}} \vec{x}^d_k = 0 & \text{if } k \in \mathcal{P} & \\
& \vec{x}_k - \sum_{d \in \mathcal{D}} \vec{x}^d_k = 0 & \text{if } k \notin \mathcal{P} & (26)
\end{array}$$

$$
\begin{array}{rll}
-\sum_{d\in\mathcal{D}} \bar{\mathbf{z}}^d \cdot \vec{\mathbf{x}}^d = 0 & & (27) \\
\vec{\mathbf{x}}^d + \vec{y}\,\bar{\mathbf{x}}^d \geq \mathbf{0} & \forall\ d \in \mathcal{D} & (28) \\
\vec{u}^d_d = 0 & \forall\ d \in \mathcal{D} & \\
-\vec{u}^d_s \geq 0 & \forall\ s \in \mathcal{O}^d & (29) \\
\vec{u}^d_i \text{ unrestricted in sign} & \text{for } i \notin \mathcal{O}^d, i \neq t &
\end{array}
$$

As discussed above, we will solve the dual (FD-D) of the direction finding linear program (FD-P). The dual variables are α^d, β^d, ϕ, ϵ, γ^d, δ^d, corresponding to constraint sets (24), (25), (26), (27), (28), (29), respectively. The dual linear program is

(FD-D)

$$
\begin{array}{rll}
\max & \sum_{k\notin\mathcal{P}\cup\mathcal{L}} \sum_{d\in\mathcal{D}} \alpha^d_k & \\
\text{subject to} & -\phi + \mathbf{A}^T\beta^d + \gamma^d - \epsilon\bar{\mathbf{z}}^d = \mathbf{0} & \forall\ d \in \mathcal{D} \\
& \mathbf{A}\alpha^d - \delta^d = \mathbf{0} & \forall\ d \in \mathcal{D} \\
& \phi_k - \frac{\partial f_k(\bar{x}_k)}{\partial x_k} \sum_{d\in\mathcal{D}} \alpha^d_k = 0 & \text{if } k \notin \mathcal{P} \\
& \sum_{d\in\mathcal{D}} \left[\bar{\mathbf{z}}^d \cdot \alpha^d + \bar{\mathbf{x}}^d \cdot \gamma^d\right] = 0 & \\
& \alpha^d \geq \mathbf{0} & \forall\ d \in \mathcal{D} \\
& \gamma^d \geq \mathbf{0} & \forall\ d \in \mathcal{D} \\
& \delta^d_d \text{ unrestricted in sign} & \forall\ d \in \mathcal{D} \\
& \delta^d_s \geq 0 & \forall\ s \in \mathcal{O}^d \\
& \delta^d_i = 0 & \text{for } i \notin \mathcal{O}^d, i \neq d.
\end{array}
$$

Each α^d_k found to be positive implies that the corresponding constraint is a member of $P^=$ and that the corresponding link k is a member of $\mathcal{P}$. The value of these nonlinear cost functions on the feasible set is therefore known and they are replaced in the formulation by their constant values. Thus at each iteration of Kerzner's algorithm we will eliminate all of the nonlinear constraints corresponding to at least one link and frequently to many links.

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